J Glob Optim (2008) 40:339–352
DOI 10.1007/s10898-007-9209-2

Interactive decisions and potential games

Lucia Pusillo

Received: 24 June 2007 / Accepted: 28 June 2007 / Published online: 26 September 2007

Abstract The aim of this contribution is an overview on Potential Games. This class of games is special, in fact we can investigate their properties by a unique function: the potential function. We consider several types of potential games: exact, ordinal, bayesian and hierarchical. Some results are generalized to multicriteria decisions.

Keywords Non cooperative games · Nash equilibrium · Potential games

AMS Classification 91A10

1 Introduction

The aim of this contribution is in the field of mathematical Theory of Games with a special overview about a class of non cooperative games: Potential Games.

(For an introduction to Game Theory see for example [19] [24]). This is a very special class of games, they have equilibria in pure strategies and several properties can be studied through their potential function.

It was in 1996 that Monderer and Shapley [17] introduced several classes of potential games.

A common property of these classes is the existence of a function (the so called potential function) which incorporates informations about the interactive decisions of all players.

First we speak about exact potential games and ordinal potential ones. The exact potential games are characterized by a function defined on the strategy space, which exactly measures the difference in payoff of each player in his own strategy. In the class of ordinal potential games, the preferences of players have a role.

L. Pusillo (✉)
Department of Mathematics, University of Genoa, Genoa, Italy
e-mail: pusillo@dima.unige.it

The first who used potential functions for games was Rosenthal [23] in 1973; he defined the class of congestion games and proved that every game in this class has a Nash equilibrium (NE for short).

In 1996 Monderer and Shapley showed that the class of games of Rosenthal (so important for economic applications) coincides (through an isomorphism) with the class of finite potential games.

In this article we consider also games with potential with incomplete information (or called Bayesian potential games). Bayesian games were introduced by Harsanyi in 1967 with the papers in Management Science [9].

Another important class for economic situation is that of hierarchical potential games. In this case we also consider approximate equilibria because they have a paradoxical behaviour: the equilibrium in a hierarchical problem (Stackelberg equilibrium) is not an approximate equilibrium. In spite of all, this way to define approximate equilibria ([16]) seems to be the most natural and the unique way to have economic applications. It may be useful to note that in [3] the authors consider the problem of sharing the cost of printers, copies and faxes among the members of a department. The even division is not a good idea because it is reasonable to keep into account whether a member uses facilities and how much times each "player" uses them. To solve this problem (so actual!) the authors propose to study a model which results to be an exact potential game.

The outline of the present paper is the following: in Sect. 2 we speak about exact potential games and ordinal potential ones, in Sect. 3 we speak about games with incomplete information or Bayesian games with potential function, in Sect. 4 we study hierarchical potential games which are important for dynamic situations [1]. In Sects. 5 and 6 we give a generalization to exact potential and ordinal potential games with multi-objectives. Section 7 concludes with some suggestions for further research.

2 Exact potential games and ordinal potential games

To have easier notations we consider only games with two players but the results are valid also for every finite number of players. By $G = (X, Y, f, g)$ we denote a game with two players, where X, Y are non-empty sets denoting the players' strategy spaces, $f, g : X \times Y \to \mathbb{R}$ are real valued functions representing the utility functions of the players.

The most accreditate notion of equilibrium for non cooperative games is the Nash equilibrium (NE for short) ([18]).

Definition 1 Given a game $G = (X, Y, f, g)$ a strategy profile $(x^*, y^*) \in X \times Y$ is said a Nash equilibrium if the two following inequalities are valid:

$$1) \quad f(x^*, y^*) \geq f(x, y^*) \tag{1}$$

$$2) \quad g(x^*, y^*) \geq g(x^*, y) \tag{2}$$

for each $x \in X$ and $y \in Y$

Monderer and Shapley introduced the idea of potential games ([17]). There are many classes of potential games and they have a common property: the existence of a real function P which simultaneously gives us informations about players possibilities.

Firstly we consider exact potential games:

Definition 2 A game $G = (X, Y, f, g)$ is said an exact potential game if there exists a function P s.t.

$$f(x_1, y) - f(x_2, y) = P(x_1, y) - P(x_2, y) \tag{3}$$

$$g(x, y_1) - g(x, y_2) = P(x, y_1) - P(x, y_2) \tag{4}$$

$\forall x, x_1, x_2 \in X$ and $\forall y, y_1, y_2 \in Y$.

The function P is called an exact potential function for G.

Example 1 (Prisonner Dilemma)

	C	D
A	33	05
B	50	11

for example $P =$

	C	D
A	3	5
B	5	6

If G is an exact potential game then it has the same NE of the pure coordination game $G^P = (X, Y, P, P)$.

Every finite game (that is with X, Y finite sets), has at least a NE which coincides with the maximum of potential function.

It is obvious that two potentials differ through a constant so in the previous example, if $c \in \mathbb{R}$ all potential functions are:

C	C+2
C−2	C+3

$\mathrm{C} \in \mathbb{R}$

Definition 3 A game $G = (X, Y, f, g)$ is a:

– coordination game if $f(x, y) = g(x, y) = P(x, y)$
– dummy game if there are two functions $h : Y \to \mathbb{R}$, $k : X \to \mathbb{R}$ s.t. $f(x, y) = h(y)$ and $g(x, y) = k(x)$.

Theorem 1 *Let G be a strategic game. G is an exact potential game if and only if* $G = G_c + G_d$ *where* G_c *is a pure coordination game and* G_d *is a dummy game.*

(See [17] and for an alternative interesting proof see [26]).

Now we consider ordinal potential games.

Definition 4 A strategic game $G = (X, Y, f, g)$ is an ordinal potential game if there exists an ordinal potential function P_0 such that:

$$f(x_1, y) > f(x_2, y) \Leftrightarrow P_0(x_1, y) > P_0(x_2, y) \tag{5}$$

$$g(x, y_1) > g(x, y_2) \Leftrightarrow P_0(x, y_1) > P_0(x, y_2) \tag{6}$$

It is easy to see that if G has an ordinal potential function P_0 then the two games G and $G^P = (X, Y, P, P)$ have the same NE and if the strategy spaces are finite then G has at least a NE.

Sum of ordinal potential games may not be an ordinal potential game (in a different way as it was for exact potential game).

3 Bayesian potential games

In this section we consider potential games with incomplete information in which a player can be of many types and he has a goal for each type.

These games arise from real life: for example in auctions the valuation of an object (a painting, a chinese vase,...), in many oligopolistic situations the cost functions of the opponents are not known, so to capture these kind of economic situations Bayesian games or games with incomplete information are very useful. J. Harsanyi with his article [9] is the founder of this interesting theory. (See also [7] for this topic.)

Now let us define a Bayesian game (with common prior).

Definition 5 A two- players Bayesian game with common prior is the t-uple:

$$G = (A_1, A_2, T_1, T_2, p, u_1, u_2)$$

For player i A_i is the action space, T_i is the types space, p is the belief; a strategy for a player is a map $x_i : T_i \to A_i$.

$u_i : A \times T \to \mathbb{R}$ is the payoff function for i.

If p is common prior on $T = T_1 \times T_2$ then

$$p_i(t_{-i}|t_i) = \frac{p(t_{-i}, t_i)}{p(t_i)} \tag{7}$$

describes the uncertainty of player i of type t_i about the type profiles t_{-i} of the opponents.

A bayesian equilibrium (BE for short) is defined as follows:

Definition 6 A strategy profile $x = (x_1, x_2) \in X_1 \times X_2$ is a Bayesian equilibrium for the game G if for all $i = 1, 2$, $t_i \in T_i$ and $a_i \in A_i$

$$\sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) u_i(\{x_j(t_j)\}_{j \in N}, t) \geq \sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) u_i(\{x_j(t_j)\}_{j \in N \setminus \{i\}}, a_i), t) \tag{8}$$

Definition 7 Given a Bayesian game $G = (A_1, A_2, T_1, T_2, p, u_1, u_2)$ the corresponding ex-ante game is the following:

$$\hat{G} = (X_1, X_2, \hat{u_1}, \hat{u_2})$$

where $X_i = (A_i)^{T_i}$ is the strategy set for player $i = 1, 2$ and the utility function is defined so:

$$\sum_{t \in T} p(t) u_i((x_j(t_j))_{j=1,2}, t)$$

Note that the *ex-ante game* $\hat{G}$ is a potential game if the game G is a bayesian potential game with common prior.

The following theorem for a game with incomplete information is due to Harsanyi ([9])

Theorem 2 *For any Bayesian game G with common prior, the following conditions are equivalent:*

i) *x^* is a Bayesian equilibrium of G*
ii) *x^* is a Nash equilibrium of $\hat{G}$ (the ex-ante game of G).*

Definition 8 Let G be a Bayesian game, it is a Bayesian potential game if there is a function $\tilde{P} : A \times T \to \mathbb{R}$ such that the following two relations are valid for each $(a_1, a_2) \in A_1 \times A_2 = A$, $(b_1, b_2) \in A_1 \times A_2 = A$, $t \in T$:

$$1)\ u_1(a_1, a_2, t) - u_1(b_1, a_2, t) = \tilde{P}(a_1, a_2, t) - \tilde{P}(b_1, a_2, t) \tag{9}$$

$$2)\ u_2(a_1, a_2, t) - u_2(a_1, b_2, t) = \tilde{P}(a_1, a_2, t) - \tilde{P}(a_1, b_2, t) \tag{10}$$

$\tilde{P}$ is called potential function of the Bayesian game G.

Remark that each bayesian potential game, with common prior, has an equilibrium in pure strategies and to determine the Bayesian equilibria of G we can consider the game $G^{\tilde{P}}$ where the utility functions are replaced by the potential function.

Generalizing the model in [23] we can define a bayesian congestion situation and the corresponding bayesian congestion game to conclude that:

Theorem 3 *If* $G = (A_1, A_2, T_1, T_2, p, u_1, u_2)$ *is a bayesian game corresponding to a bayesian congestion model, then G is a bayesian potential game.*

It is not always true that a bayesian potential game arises from a bayesian congestion situation as proven in [8].

Let us see an example of bayesian potential game.

Example 2 Let us consider the following bayesian game with two players:
$G = (A_1, A_2, T_1, T_2, p_1, p_2, u_1, u_2)$ where
$A_1 = \{A, B\}$, $A_2 = \{C, D\}$, $T_1 = \{t_1, t_2\}$, $T_2 = \{s_1, s_2\}$, further $p_1(t_1, s_2) = p_1(t_1, s_1) = p_1(t_2, s_2) = p_1(t_2, s_1) = 1/4$
$p_2(t_1, s_1) = p_2(t_2, s_2) = 1/2$
$p_2(t_1, s_2) = p_2(t_2, s_1) = 0$

The strategies of players are:

$X_1 = \{AA, AB, BA, BB\}$, $X_2 = \{CC, CD, DC, DD\}$ where the first letter says what player I makes if he is of type t_1 and the second letter if he is of type t_2 (analogously for player II if he is of type s_1 or of type s_2).

For each pair of types the corresponding bimatrix game is an exact potential game so we call it a bayesian potential game.

The matrix of players' payoff is in the following table.

		s_1		s_2	
		C	D	C	D
t_1	A	1, 1	2, 3	2, 1	2, 1
	B	3, 2	1, 1	2, 1	2, 1
t_2	A	2, 1	2, 1	2, 2	4, 3
	B	1, 2	1, 2	4, 5	3, 3

A potential for this game is:

t_1

s_1

	C	D
A	2	1
B	2	1

s_2

	C	D
A	1	2
B	1	2

t_2

s_1

	C	D
A	1	1
B	0	0

s_2

	C	D
A	1	2
B	3	1

4 Hierarchical potential game

In this section we introduce the Stackelberg problem [25] which is a hierarchic situation. Further we see how hierarchical potential games, which are a special class of potential games, are suitable to describe dynamical situations.

In this case we consider also approximate equilibria because they have a paradoxical behaviour as we are going to see (for details see [14]).

Definition 9 Let $G = (X, Y, f, g)$ be a game and consider the following problem: find $\overline{x} \in X$ such that

$$\inf_{y \in R_{II(\overline{x})}} f(\overline{x}, y) \geq \inf_{y \in R_{II(x)}} f(x, y) \quad \forall x \in X \tag{11}$$

where $R_{II(\overline{x})} = \text{argmax}_{y \in Y}\, g(x, y)$.

A pair $(\overline{x}, \overline{y}) \in X \times Y$ with $\overline{x}$ satisfying (11) and $\overline{y} \in R_{II(\overline{x})}$ is called pessimistic Stackelberg equilibrium (for short pSE) and $\overline{x}$ is called a pessimistic Stackelberg solution (or weak Stackelberg solution).

If we write "sup" instead of "inf", we have the optimistic Stackelberg solution (for short oSE) or strong Stackelberg solution.

If we let $\beta(x) = \inf_{y \in R_{II(x)}} f(x, y)$ and $\gamma(x) = \sup_{y \in R_{II(x)}} f(x, y)$ then $(\overline{x}, \overline{y}) \in X \times Y$ is pSE if it satisfies

$\beta(\overline{x}) = \max \beta(x)$ and $\overline{y} \in R_{II(\overline{x})}$ and $(\tilde{x}, \tilde{y}) \in X \times Y$ is oSE if it verifies

$\gamma(\tilde{x}) = \max \gamma(x)$ and $\tilde{y} \in R_{II(\tilde{x})}$.

There is no link between these two definitions, as we show by the following example.

Example 3 Let G be a game where $X = Y = [0, 1]$,

$f(x, y) = 3x + 6y - 4xy, \quad g(x, y) = \max\{y, 1/2\}$,

$A = \{(0, y) : \ y \in [1/2, 1]\}$ is the set of optimistic Stackelberg equilibria,

$B = \{(1, y) : \ y \in [1/2, 1]\}$ is the set of pessimistic Stackelberg equilibria.

If we consider $X = Y = (0, 1]$ the set of oSE is empty and the set of pSE is not empty.

If we consider $X = Y = [0, 1)$ the set of pSE is empty and the set of oSE is not empty.

Let us define the approximate Stackelberg equilibria (see [16] and references in it).

Definition 10 Given $(\epsilon, \eta) \in \mathbb{R}^2$ with $\epsilon, \eta \geq 0$, $\overline{x} \in X$ is an (ϵ, η) pessimistic Stackelberg solution to problem (11) if, $\forall x \in X$

$$\inf_{y \in R_{II(x,\eta)}} f(x, y) - \inf_{y \in R_{II(\overline{x},\eta)}} f(\overline{x}, y) \leq \epsilon \tag{12}$$

where $R_{II(x,\eta)} = \{\tilde{y} \in Y : \; g(x, y) - g(x, \tilde{y}) \leq \eta \quad \forall y \in Y\}$ that is if player I is unlucky, he does not lose more than ϵ.

We say that $(\overline{x}, \overline{y})$ is a pessimistic $S(\epsilon, \eta)$ if it satisfies the condition (12) and $\overline{y} \in R_{II(x,\eta)}$. We write optimistic $S(\epsilon, \eta)$ for the set of elements satisfying condition (12) with "sup" instead of "inf".

Definition 11 $G = (X, Y, f, g)$ is a hierarchical potential game with two players if there is a potential function $P : X \times Y \to \mathbb{R}$ and $h : X \to \mathbb{R}$ s.t.

$$\begin{cases} f(x, y) = P(x, y) \\ g(x, y) = P(x, y) + h(x) \end{cases}$$

So a hierarchical potential game is a particular case of an exact potential game.

Proposition 1 *The following conditions are equivalent for a hierarchical potential game with potential P:*

i) $(\overline{x}, \overline{y}) \in X \times Y$ *is a pessimistic Stackelberg equilibrium* (pSE) *of* G
ii) $(\overline{x}, \overline{y}) \in X \times Y$ *is a pessimistic Stackelberg equilibrium* (pSE) *of* G^P
iii) $(\overline{x}, \overline{y}) \in argmax_{(x,y)\in X\times Y)} P(x, y)$
iv) $(\overline{x}, \overline{y}) \in X \times Y$ *is a optimistic Stackelberg equilibrium* (oSE) *of* G
v) $(\overline{x}, \overline{y}) \in X \times Y$ *is a optimistic Stackelberg equilibrium* (oSE) *of* G^P

For a proof see [15].

The sets of approximate Stackelberg equilibria do not coincide in the case of optimistic and pessimistic one and they do not coincide as well as with the superlevels of the potential function P as the following Lemma and example show.

Lemma 1 *Let* $M = sup_{X\times Y} P(x, y) < +\infty$ *and let* $L(\epsilon) = \{(x, y) \in \mathbb{R}^2 : \mathrm{P(x, y)} \geq \mathrm{M} - \epsilon\}$*, that is the set of superlevels of P. The following inclusions are valid:*

i) $L(\epsilon) \subset (\epsilon + \eta, \eta)oSE \quad if \;\; \epsilon \leq \eta$
ii) $(\epsilon, \eta)oSE \subset (\epsilon + \eta, \eta)pSE$
iii) $(\epsilon, \eta)pSE \subset L(2\epsilon + \eta)$.

For proof see [14].

Example 4 Let $G = (\mathbb{R}, \mathbb{R}, \mathrm{f}, \mathrm{g})$ be a game with
$f(x, y) = g(x, y) = -x^2 - y^2 = P(x, y)$
$L(\delta) = \{(x, y) \in [-1, 1]^2 \text{ s. t. } x^2 + y^2 \leq \delta\}$
$L(\delta) \subset (\epsilon, \eta)oSE = (\epsilon, \eta)pSE = [-\sqrt{\epsilon}, \sqrt{\epsilon}] \times [-\sqrt{\eta}, \sqrt{\eta}]$ if $\delta = \min(\epsilon, \eta)$.
Furthermore
$(\epsilon, \eta)pSE \subset L(\delta_1)$ if $\delta_1 = \epsilon + \eta$.

In the following example we see a paradoxiacal situation: the Stackelberg equilibrium is not an approximate equilibrium (we see this fact for pessimistic Stackelberg equilibrium but in a similar way, we can do an example for optimistic one).

Example 5 Let be $G = ((0,1], (0,1], f, g)$ with
$f(x, y) = -x - y + 2xy, g(x, y) = y^2 - y$
$R_{II}(x) = argmax_{y\in(0,1]} g(x, y) = 1.$
Then $R_{II}(x, \eta) = \{y : g(x, y) \geq \max_{y\in(0,1]} g(x, y) - \eta\} = (0, \frac{1-\sqrt{1-4\eta}}{2}] \cup [\frac{1+\sqrt{1-4\eta}}{2}, 1]$ if $\eta < 1/4$.
$(\epsilon, \eta)pSE = \{(x, y) : x \in [1/2 - \epsilon, 1/2 + \epsilon], y \in R_{II}(x, \eta)\}$, if $\eta < 1/4, \epsilon < 1/2$.
The unique $pSE : (1, 1) \notin (\epsilon, \eta)pSE$.

$$\gamma(x, \eta) = \sup_{y\in R_{II}(x,\eta)} f(x, y) = \begin{cases} x - 1 & \text{if } x \geq 1/2 \\ -x & \text{if } x < 1/2 \end{cases}$$

$(\epsilon, \eta)oSE =$
$\{(x, y) : x \in (0, \epsilon] \cup [1 - \epsilon, 1], y \in R_{II}(x, \eta)\}$.
The unique $oSE : (1, 1) \in (\epsilon, \eta)oSE$.

Remark 1 The approximate Stackelberg equilibria in general are increasing sets with respect to ϵ but for hierarchical potential games the optimistic approximate sets are increasing with respect to ϵ and η.

An interesting example of hierarchical potential game arising from a sequential production situation (which has practical motivations) is a game based on the processing of rough diamonds (see [27]).

In this paper the authors prove that a hierarchical game is a hierarchical potential game if and only if its normalization is an exact potential game.

5 A generalization: multi-criteria potential games

In a lot of problems we must maximize several criteria, in fact decisions are guided by multiple goals, often not comparable.

So multicriteria games are a natural extension of one criterium game: every strategic game can be thought as a multicriteria game where each player has one criterium to maximize. In this paper we consider only bicriteria games for easier notations, but all results are valid also for a finite number of criteria. Let us give the formal definition:

Definition 12 A non cooperative bicriteria game is a quadruplet $G = (X_1, X_2, u_1, u_2)$ where X_1, X_2 are the pure strategy sets for the two players, respectively of player I and of player II.

The utility functions are: $u_i : X_1 \times X_2 \to \mathbb{R}^2$i $= 1, 2$, that is each player wishes to maximize not one but two criteria.

To show in a better way the situation let us consider the following example of a bicriteria game from the book of Voorneveld ([26]):

Example 6 Let us consider a game G with two players: an inspector and a factory. The first player must decide if inspect a factory to see if its products are hygienical or not so he must minimize the inspections costs and to guarantee a good level of hygiene . The second player must minimize the production costs and to have an accettable level of hygiene in his production. The inspector's strategies are: A (ispection) and B (no inspection) and factory's one are C (hygienical) and D (no hygienical). The number c ($c > 1$) denote the penalty which must be imposed if the inspection shows a not hygienic production. For the inspector the first coordinate means the cost of inspection and the second coordinate shows the satisfaction for an hygienical situation.

For the factory, the first coordinate shows the negative costs of productions, the second one the satisfaction.

	C	D
A	$-1, 1$	$c-1, 1/2$
B	$0, 1$	$0, 0$

inspector's payoff

	C	D
A	$-1, 1$	$-c-1, 1$
B	$-1, 1$	$0, 0$

factory's payoff

Definition 13 Let $G = (X_1, X_2, u_1, u_2)$ a bicriteria game. A strategy profile $(x_1, x_2) \in X_1 \times X_2$ is a

1) weak Pareto equilibrium (wPE for short) if

$$\nexists y_1 \in X_1 : u_1(y_1, x_2) > u_1(x_1, x_2) \tag{13}$$

and

$$\nexists y_2 \in X_2 : u_2(x_1, y_2) > u_2(x_1, x_2) \tag{14}$$

2) strong Pareto equilibrium (sPE for short) if

$$\nexists y_1 \in X_1 : u_1(y_1, x_2) \geq u_1(x_1, x_2) \tag{15}$$

and

$$\nexists y_2 \in X_2 : u_2(x_1, y_2) \geq u_2(x_1, x_2) \tag{16}$$

In all the paper we consider only pure strategies.

Definition 14 A bicriteria game $G = (X_1, X_2, u_1, u_2)$ is said:

– pure coordination game if there exists a function $u : X_1 \times X_2 \to \mathbb{R}^2$ s.t. $u_1 = u_2 = u$
– dummy game if there are two functions $k : X_1 \to \mathbb{R}^2$, $h : X_2 \to \mathbb{R}^2$ s.t. $u_1(x_1, x_2) = k(x_1)\ \forall x_1 \in X_1$ and $u_2(x_1, x_2) = h(x_2)\ \ \forall x_2 \in \mathbb{R}^2$

In a pure coordination game, the players have the same payoffs, in a dummy game the payoff of one player depends only by the strategies of the other.

(0, 3) (0, 3)	(1, 1) (1, 1)
(0, 0) (0, 0)	(2, 1) (2, 1)

Bicriteria coordination game

(0, 0) (0, 2)	(3, 4) (0, 2)
(0, 0) (1, 3)	(3, 4) (1, 3)

Bicriteria dummy game.

Definition 15 A bicriteria game $G = (X_1, X_2, u_1, u_2)$ is an exact potential bicriteria game if there exists a function $P : X_1 \times X_2 \to \mathbb{R}^2$ (called potential function) s.t.

$$u_1(x_1, x_2) - u_1(y_1, x_2) = P(x_1, x_2) - P(y_1, x_2) \tag{17}$$

and

$$u_2(x_1, x_2) - u_2(x_1, y_2) = P(x_1, x_2) - P_2(x_1, y_2) \tag{18}$$

$\forall x_1, y_1 \in X_1, x_2, y_2 \in X_2$

Remark 2 Given the following bicriteria game in strategic form:

$(a, b)\ (c, d)$	$(e, f)\ (g, h)$
$(l, m)\ (n, o)$	$(p, q)\ (r, s)$

G is an exact potential game if the following equality is true:

$$(g, h) - (c, d) + (p, q) - (e, f) + (n, o) - (r, s) + (a, b) - (l, m) = 0$$

Example 7 The following is an exact potential bicriteria game in strategic form: and P a Potential function:

(0, 3) (0, 5)	(4, 5) (1, 3)
(0 0) (1, 3)	(5, 5) (3, 4)

Exact potential bicriteria game

P :

0, 5	1, 3
0, 2	2, 3

All potential functions of this game are:

P_{hk} :

$(h,\ k)$	$(h+1,\ k-2)$
$(h,\ k-3)$	$(h+2,\ k-2)$

$(h, k) \in \mathbb{R}^2$

The following example of duopoly is an exact potential game with two criteria:

Example 8 Let I,II be two firms which produce two type of mineral water, without bubbles (x) and with bubbles (y).

$x = \binom{x_1}{x_2}$ where x_i is the mineral water without bubbles for the enterprise i.

$y = \binom{y_1}{y_2}$ where y_i is the mineral water with bubbles for the enterprise i.

$Q = x_1 + y_1 + x_2 + y_2$ is the total quantity of mineral water.

The utility function are:

$$\begin{aligned}\pi_1(x, y) &= \begin{pmatrix} x_1 F(x_1 + x_2 + y_1 + y_2) - c_{11}(x_1) \\ y_1 F(x_1 + x_2 + y_1 + y_2) - c_{12}(y_1) \end{pmatrix} \\ &= \begin{pmatrix} x_1(a - b(x_1 + x_2 + y_1 + y_2)) - c_{11}(x_1) \\ y_1(a - b(x_1 + x_2 + y_1 + y_2)) - c_{12}(y_1) \end{pmatrix} \\ \pi_2(x, y) &= \begin{pmatrix} x_2 F(x_1 + x_2 + y_1 + y_2) - c_{21}(x_2) \\ y_2 F(x_1 + x_2 + y_1 + y_2) - c_{22}(y_2) \end{pmatrix} \\ &= \begin{pmatrix} x_2(a - b(x_1 + x_2 + y_1 + y_2)) - c_{21}(x_1) \\ y_2(a - b(x_1 + x_2 + y_1 + y_2)) - c_{22}(y_2) \end{pmatrix}.\end{aligned}$$

where we have written $c_{ij}(.)$ for the cost of firm i for the product j. Cost functions are differentiable. F (the inverse demand function) is linear, $F(Q) = a - bQ,\ a, b > 0$ An exact potential function for G is $P = \binom{P_1}{P_2}$ with

$$\begin{aligned}P_1(x, y) &= a(x_1 + x_2 + y_1 + y_2) - b(x_1^2 + x_2^2 + y_1^2 + y_2^2) \\ &\quad -b(x_1x_2 + x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2 + y_1y_2) - c_{11}(x_1) - c_{21}(x_2) \\ P_2(x, y) &= a(x_1 + x_2 + y_1 + y_2) - b(x_1^2 + x_2^2 + y_1^2 + y_2^2) \\ &\quad -b(x_1x_2 + x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2 + y_1y_2) - c_{12}(y_1) - c_{22}(y_2). \qquad \square\end{aligned}$$

Now let us see congestion models, introduced by Rosenthal in 1973 and we generalize them to multi-objectives.

Definition 16 A congestion model is a tuple $< N, M, (X_i)_i \in N, (c_j)_{j \in M} >$ where:

- N is the set of players $N = 1, 2$
- M is the finite set of facilities
- For each player $i \in N$, his collection of pure strategies x_i is a finite family of subsets of M
- For each facility $j \in M$, $c_j : \{1, 2\} \to \mathbb{R}^2$ is the cost function of facility j, with $c_j(r)$, $r \in \{1, 2\}$ the costs to each of the users of machine j if there is a total of r users.

This leads us to a congestion game: $G = < N, X_1, X_2, u_1, u_2 >$ where N, X_1, X_2 are defined as above and

$\forall x = (x_1, x_2) \in X_1 \times X_2$ and $\forall j \in M$ let $\eta_j(x)$ be the number of users of machine j if the players choose $x = (x_1, x_2)$ then the utility functions are $u_i(x) = u_i(x_1, x_2) = -\sum_{j \in x_i} c_j(\eta_j(x))$

This definition tells us that every player pays for the facility he uses with costs depending only on the number of users of the facility.

Usually, it is assumed that costs are increasing functions of the number of users.

Now we introduce the notion of isomorphism between two games because it will be useful to see that an exact potential bicriteria game is isomorpic to a bicriteria congetion one.

Definition 17 Let $G = (X_1, X_2, u_1, u_2)$ and $H = (Y_1, Y_2, v_1, v_2)$ be two bicriteria strategic games with the same players.

$u_i : \prod_{i=1,2} X_i \to \mathbb{R}^2$, $v_i : \prod_{i=1,2} Y_i \to \mathbb{R}^2$, G, H are isomorphic games if $\forall i = 1, 2$ there exists a bijection $\phi_i : X_i \to Y_i$ s.t. $u_i(x_1, x_2) = v_i(\phi_1(x_1), \phi_2(x_2)), \forall x_i \in X_i$.

The following example is a bicriteria congestion game.

Example 9 Let us suppose that two people must go from a town to another. They have the possibility to choose between two streets: A and B. They must minimize time and gasoline:

So the bicriteria congestion game has the following matrix:

	A	*B*
A	(10, 10) (6, 6)	(5, 8) (4, 1)
B	(8, 5) (1, 4)	(12, 12) (8, 8)

Theorem 4 *Each finite exact potential bicriteria game is isomorphic to a congestion one.* ([22])

6 Bicriteria ordinal potential games

Definition 18 A bicriteria game $G = (X_1, X_2, u_1, u_2)$ is an ordinal potential bicriteria game if there exists a function $P : X_1 \times X_2 \to \mathbb{R}^2$ (called ordinal potential function) s.t.

$$u_1(x_1, x_2) - u_1(y_1, x_2) \geq 0 \Leftrightarrow P(x_1, x_2) - P(y_1, x_2) \geq 0 \tag{19}$$

and

$$u_2(x_1, x_2) - u_2(x_1, y_2) \geq 0 \Leftrightarrow P(x_1, x_2) - P(x_1, y_2) \geq 0 \tag{20}$$

$\forall x_1, y_1 \in X_1, x_2, y_2 \in X_2$.

Example 10

(0 0) (2 0)	(0 1) (3 1)
(1 2) (0 0)	(0 0) (1 1)

This is a bicriteria ordinal potential game, in fact P_o defined as below is an ordinal potential of this game:

P_o :

0, 0	2, 3
1, 1	2, 2

Remark 3 Every order preserving trasformation of an ordinal potential function is again an ordinal potential function of the game.

Remark 4 The set of ordinal potential bicriteria game is not closed under addition. Let us consider:

G_3 :

(0, 0) (2, 0)	(0, 1) (3, 1)
(1, 2) (0, 0)	(0, 0) (1, 1)

and

G_4 :

(1, 1) (0, 2)	(1, 1) (1, 0)
(−1, −2) (3, 3)	(2, 3) (1, 1)

These games have ordinal potential respectively:

P_{03} :

0, 0	2, 3
1, 1	2, 2

P_{04} :

0, 2	1, 0
3, 9	2, 1

$G_5 = G_3 + G_4$

(1, 1) (2, 2)	(1, 2) (4, 1)
(0, 0) (3, 3)	(2, 3) (2, 2)

which has not ordinal potential: it has not weak improvement cycle. We can generalize to multicriteria games the properties about the improvement of cycles given for one criterium in [17].

Let us see a duopoly bicriteria game which turns out to be an ordinal potential game:

Example 11 The problem is a Cournot game with two objectives, it is similar to example 10, with the same notations but the inverse demand function $F(Q)$ with $Q > 0$ is a positive function without other hypotheses. We suppose that cost functions are linear and the costs of mineral water (of the same type) is equal for the two firms; we write: $c_{11} = c_{21} = c_\ell$ and $c_{12} = c_{22} = c_g$

Let us define a function $P : \mathbb{R}^2_+ \times \mathbb{R}^2_+ \to \mathbb{R}^2$ in the following way:

$$P(x, y) = \begin{pmatrix} x_1x_2(F(x_1 + x_2 + y_1 + y_2) - c_\ell \\ y_1y_2(F(x_1 + x_2 + y_1 + y_2) - c_g \end{pmatrix}. \tag{21}$$

This potential function P is an ordinal potential for the bicriteria Cournot game so we conclude that the game described is an ordinal potential game.

7 Concluding remarks and open problem

In this article we have considered several classes of potential games: exact, ordinal, hierarchical, bayesian and we have presented some possible generalizations to multicriteria games.

Our choice, as already said, was determined by the fact that this class of games have Nash equilibria in pure strategies and they have a potential function (respectively exact, ordinal, hierarchical, bayesian) which gives informations about strategic decisions of all players.

There are still many open problems which can be investigated:

1) what can we say about a generalization from bayesian potential and hierarchical potential games to multicriteria ones?
2) In previous papers [11,13,14,20,21] Tihkonov well posedness property was studied for non cooperative games. What about the well posedness property for bayesian games? and for multicriteria?
3) We have seen that approximate equilibria have a strange behaviour in the hierarchical problems for the corresponding potential games. What can we say about approximate Pareto equilibria for multicriteria games?
4) In [2] were introduced supermodular games which, under some hypotheses, are "close" to potential games. What can we say about a new class of bayesian supermodular games?
5) We have seen that the congestion games arising from a congestion situation are potential games. Some traffic problems studied in [4–6,10] can be reconsidered from the point of view of congestion models and so studied through potential games?

Some of these topics are in progress.

Acknowledgements The author is gratefully indebted with Prof. F. Patrone and Prof. S. Tijs for many interesting discussions.

References

1. Basar, T., Olsder, G.J.: Dynamic Noncooperative Games. Academic Press, New York (1982)
2. Branzei, R., Mallozzi, L., Tijs, S.: Supermodular games and potential games. J. Math. Econ. **39**, 39–49 (2003)
3. Briata, F., Moretti, S., Patrone, F., Tijs, S.: Printers: Non cooperative games from Tuic games, preprint (2006)
4. Daniele, P., Maugeri, A.: Vector variational inequalities and a continuum modelling of traffic equilibrium problem. In: Giannessi, F. (ed.) Vector Variational Inequalities and Vector Equilibria, pp. 97–112. Kluwer Academic Publishers (2000)
5. Daniele, P., Maugeri, A., Oettli, W.: Variational Inequalities and time dependent Traffic Equilibria C.R. Acad. Sci. Paris, 326, serie I (1998)
6. Daniele, P., Maugeri, A., Oettli, W.: Time dependent traffic equilibria. J. Optim. Th. Appl. **103**(3), 543–555 (1999)
7. Facchini, G., Van Megen, F., Borm, P., Tijs, S.: Congestion models and weighted bayesian potential games. Theory and Decision **42**, 193–206 (1997)
8. van Heumann, R., Peleg, B., Tijs, S., Borm, P.: Characterization of solutions for Bayesian games. Theory and Decision **40**, 103–129 (1996)
9. Harsanyi, J.: Games with incomplete information played by bayesian players. Manage. Sci. **14**, 159–182, 320–334, 486–502 (1967–1968)
10. Maugeri, A.: New classes of variational inequalities and applications to equilibrium problems. Methods Oper. Res. **53**, 129–131 (1985)
11. Margiocco, M., Patrone, F., Pusillo, L.: On Tihkonov well-posedness of concave games and Cournot oligopoly. J. Optim. Theory Appl. **112**, 361–369 (2002)

12. Margiocco, M., Pusillo, L.: (ϵ, k) Equilibria and Well Posedness. Int. Game Theory Rev. **8**(1), 33–44 (2006)
13. Margiocco, M., Pusillo, L.: Potential Games and Well-posedness properties. Optimization (to appear)
14. Margiocco, M., Pusillo, L.: Stackelberg Well-Posedness and Hierarchical Potential Games. Anna. Dyna. Games (to appear)
15. Mallozzi, L., Tijs, S., Voorneveld, M.: Infinite Hierarchical potential games. USSR J. Comp. Math. Math. Phys. J. Optimi. Theory Appl. **107**, 287–296 (2000)
16. Morgan, J.: Constrained well-posed two-level optimization problems. In: Clarke, F.H., Dem'yanov, V.F., Giannessi, F. (eds.) Nonsmooth Optimization and Related Topics, pp. 307–325. Plenum Press, New York (1989)
17. Monderer, P., Shapley, L.: Potential games. Games Econ. Behav. **14**, 124–143 (1996)
18. Nash, J.F. Jr.: Equilibrium points in n-person games. Proc. Nat. Acad. Sci. USA **36**, 48–49 (1950)
19. Patrone, F.: Decisori (razionali) interagenti.Una introduzione alla Teoria dei Giochi, ed. University Press, Pisa (2006)
20. Pusillo, L.: Approximate Solutions and Tikhonov Well-posedness for Nash Equilibria. In: Giannessi, F., Maugeri, A., Pardalos, P. (eds.) Equilibrium Problems:Non smooth Optimization and Variational Inequality Models, pp. 231–244. Kluwer Academic Publishers (2001)
21. Pusillo L.: Well posedness and optimization problems. In: Giannessi, F., Maugeri, A. (eds.) Variational Analysis and Applications. Springer (2005)
22. Pusillo, L., Tijs, S.: Multicriteria Games and Potentials, preprint D.I.M.A. n.524 (2005)
23. Rosenthal, R.W.: A class of games possessing pure strategy Nash equilibria. Int. J. Game Theory **2**, 65–67 (1973)
24. Tijs S.: Introduction to Game Theory. Hindustan Book Agency (2003)
25. von Stackelberg, H.: Marktform und Gleichgewitch. Springer, Vienna (1934)
26. Voorneveld, M.: Potential games and interactive decisions with multiple criteria. Ph.D. thesis CentER, Tilburg University (1999)
27. Voorneveld M., Tijs S., Mallozzi L. (1999) Sequential production situations and potentials. In: Patrone, F., Garcia Jurado, I., Tijs, S. (eds.) Game Practice Book. Kluwer Academic Publishers, Boston (1997)